

PERSISTENCE UNDER WEAK DISORDER OF AC SPECTRA OF QUASI-PERIODIC SCHRÖDINGER OPERATORS ON TREES GRAPHS

MICHAEL AIZENMAN AND SIMONE WARZEL

ABSTRACT. We consider radial tree extensions of one-dimensional quasi-periodic Schrödinger operators and establish the stability of their absolutely continuous spectra under weak but extensive perturbations by a random potential. The sufficiency criterion for that is the existence of Bloch-Floquet states for the one dimensional operator corresponding to the radial problem.

Dedicated to Ya. Sinai on the occasion of his seventieth birthday

Keywords: Random operators, absolutely continuous spectrum, quasi-periodic cocycles, Bloch states (2000 Mathematics Subject Classification: 47B80, 37E10)

CONTENTS

1. Introduction	1
2. Quasi-periodic Schrödinger operators on rooted tree graphs	3
2.1. Remarks on the AC spectrum	3
2.2. A criterion for the stability of the AC spectrum	4
3. Bloch-Floquet eigenfunctions and the main result	5
4. Uniqueness of the covariant solution of the projective Schrödinger equation	6
4.1. BF states and reducibility of the Schrödinger cocycle	6
4.2. Reducibility and uniqueness	6
5. Proof of the main result	7
Acknowledgement	7
References	7

1. INTRODUCTION

In this work we consider the effects of weak disorder on the absolutely continuous (AC) spectra of tree extensions of one-dimensional Schrödinger operators $H(\theta)$ with quasi-periodic (QP) potential. It is shown that contrary to the corresponding one-dimensional problem, under certain conditions AC spectra on trees persist under perturbations by weak random potentials.

The operator $H(\theta)$ is of the form

$$H(\theta) \psi_n := \psi_{n+1} + \psi_{n-1} + U(S^n \theta) \psi_n, \quad (1.1)$$

Date: Submitted (Mosc. Math. J.): 14 April 2005; Revised: 22 March 2006.

where $U : \Xi \rightarrow \mathbb{R}$ is a continuous function on a multidimensional torus $\Xi \equiv [0, 2\pi)^\nu$ and S is the shift

$$S\theta := (\theta + 2\pi\alpha) \bmod 2\pi. \quad (1.2)$$

with a frequency vector $\alpha = (\alpha_1, \dots, \alpha_\nu)$ for which the action of S is ergodic.

To $H(\theta)$ one may associate a “fanned-out” radial operator $\widehat{H}(\theta)$ which acts on the Hilbert space $\ell^2(\mathbb{T})$ over the vertex set \mathbb{T} of a rooted regular tree graph as

$$\widehat{H}(\theta) \psi_x := \sum_{y \in \mathcal{N}_x} \psi_y + \sqrt{K} U(S^{|x|}\theta) \psi_x. \quad (1.3)$$

In the above expression, the sum ranges over the set \mathcal{N}_x of neighboring vertices of $x \in \mathbb{T}$, the distance to the root is denoted by $|x|$, and $K \geq 2$ is the branching number of the regular tree, i.e., the number of forward neighbors of each vertex. Next, disorder is added in the form of a random potential, which is given by a collection of independent and identically distributed (iid) random variables $\omega := \{\omega_x\}_{x \in \mathbb{T}}$ associated with the vertices of \mathbb{T} . In this fashion, one obtains the ergodic operator

$$\widehat{H}_\lambda(\theta, \omega) \psi_x = \widehat{H}(\theta) \psi_x + \lambda \omega_x \psi_x \quad (1.4)$$

where $\lambda \in \mathbb{R}$ is the strength of the perturbation.

One dimensional Schrödinger operators can be studied by means of the associated cocycle, which in its projective representation is the skew product transformation $(S, \mathcal{A}(E, \cdot))$ mapping the product space $\Xi \times \mathbb{C}^+$ as follows

$$(\theta, \gamma) \mapsto (S\theta, \mathcal{A}(E, \theta)\gamma) \quad (1.5)$$

where $\mathcal{A}(E, \theta)$ is the Möbius transformation

$$\mathcal{A}(E, \theta) : \gamma \mapsto U(\theta) - E - \frac{1}{\gamma}. \quad (1.6)$$

The issue of the stability of the AC spectra of ergodic radial potentials on trees was addressed in the recent work [1]. It was shown there that AC spectra do not disappear under weak disorder if for almost every $E \in \sigma_{\text{ac}}(H(\theta))$ there is at most one (measurable) function $\Gamma : \Xi \rightarrow \mathbb{C}^+$ whose graph is invariant under the action of the Schrödinger cocycle, i.e., for which

$$\mathcal{A}(E, \theta) \Gamma(\theta) = \Gamma(S\theta). \quad (1.7)$$

As is explained below, solutions of (1.7) in \mathbb{C}^+ are directly associated with covariant eigenstates of $H(\theta)$. For energies $E \in \sigma_{\text{ac}}(H(\theta))$ one pair of conjugate eigenstates is generally obtained through the Green function, or alternatively the Weyl-Titchmarsh function (a generalization of this function to trees is discussed in a related context in [2]).

Here, we show that for operators with QP potentials the above condition – uniqueness of solutions of (1.7) – is satisfied whenever $H(\theta)$ admits a pair of Bloch-Floquet eigenstates with Bloch momenta outside of the countable collection of resonant values corresponding to the spectrum of the shift operator S . The existence of BF states at almost all energies in the AC spectrum is well known to be related to the issue of reducibility of the Schrödinger cocycle to a constant [11, 20, 21]. Such reducibility plays also an essential role in our main result.

The theory of Bloch-Floquet states for QP operators is a topic which has been strongly developed since the early works by Dinaburg and Sinai [10]. Their existence, which is required for our results, was established by KAM and duality methods for a variety of

cases [10, 5, 23, 7, 11, 12, 13, 6, 21], culminating in the recent proof by Avila and Krikorian [3] for which the sufficiency assumptions are just a Diophantine condition on α and smoothness (C^∞) of the potential.

2. QUASI-PERIODIC SCHRÖDINGER OPERATORS ON ROOTED TREE GRAPHS

2.1. Remarks on the AC spectrum. Without the disorder, the AC spectrum of the fanned-out operator $\widehat{H}(\theta)$ coincides with that of $H(\theta)$, and both can be characterized in term of the latter's Lyapunov exponent $\gamma(E)$ (see [4] for its definition):

Proposition 2.1. *For any QP potential and any θ*

$$\sigma_{\text{ac}}(\widehat{H}(\theta)) = \sqrt{K} \sigma_{\text{ac}}(H(\theta)) = \sqrt{K} \Sigma_{\text{ac}}, \quad (2.1)$$

where Σ_{ac} is the (Lebesgue) essential closure of the set $\{E \in \mathbb{R} : \gamma(E) = 0\}$.

Proof. It was noted in [1, Prop. A.1] that $\sigma_{\text{ac}}(\widehat{H}(\theta)) = \sqrt{K} \sigma_{\text{ac}}(H^+(\theta))$, where $H^+(\theta)$ is the restriction of (1.1) to the half line. For the latter, Kotani theory and the statement of independence from θ proven in [17, Thm. 6.1], ensure that $\sigma_{\text{ac}}(H^+(\theta)) = \sigma_{\text{ac}}(H(\theta)) = \Sigma_{\text{ac}}$ for all θ . \square \square

Remarks 2.2. (i) By the Kotani principle, the presence of an AC component in the spectrum of a one-dimensional Schrödinger operator requires its potential to be *deterministic* [15, 22]. For QP potentials this principle does not stand in the way of $H(\theta)$ having AC spectrum. The existence of such a component was established for various cases; an overview can be found in [18, 13, 6].

(ii) The most studied QP operator is the *almost Mathieu operator* with the potential $V_n(\theta) = u \cos(2\pi\alpha n + \theta)$, at an irrational frequency $\alpha \in (0, 1)$. In terms of (1.1) it corresponds to $\nu = 1$, i.e., $\Xi = [0, 2\pi]$, and

$$U(\theta) = u \cos(\theta). \quad (2.2)$$

As was shown in [13] for almost all α and θ the spectrum of $H(\theta)$ is

- pure absolutely continuous if $u < 2$,
- pure singular continuous if $u = 2$
- pure point with exponentially localized eigenfunctions if $u > 2$.

(iii) A broader class of potentials, still with one frequency, is obtained by letting $U(\theta) = uf(\theta)$ with a real analytic, 2π -periodic function f . For the corresponding QP Schrödinger operators it was shown [6] that for any α satisfying a Diophantine non-resonance condition $\sigma(H(\theta)) = \sigma_{\text{ac}}(H(\theta))$ provided u is small enough.

The following observation may highlight the result presented below. By the reduction of the radial spectral problem to the one dimensional case, and the above mentioned Kotani principle [15, 22], the AC spectra of the fanned-out operators are unstable under the addition of arbitrarily weak radial disorder:

Proposition 2.3. *If $\{\omega_x\}_{x \in \mathbb{T}}$ are replaced by radial iid random variables, i.e. $\omega_x = \omega_{|x|}$, for which $\mathbb{E} [\log(1 + |\omega_0|)] < \infty$, then the almost-sure AC spectrum vanishes,*

$$\sigma_{\text{ac}}(\widehat{H}_\lambda(\theta, \omega)) = \emptyset, \quad (2.3)$$

for any $\lambda \neq 0$.

The main result presented here is that the response is different when the disorder is not constrained to be radial.

2.2. A criterion for the stability of the AC spectrum. The stability criterion which was derived in [1] is expressed in terms of the uniqueness of solution of an equation associated with the Schrödinger cocycle (1.7). The equation can be viewed as describing a covariant Schrödinger eigenstate, expressed in the Ricatti form. Before explaining these concepts, let us present the criterion in an explicit form.

Proposition 2.4 ([1]). *Let $\hat{H}_\lambda(\theta, \omega) = \hat{H}(\theta) + \lambda\omega$, with $\hat{H}(\theta)$ an operator of the form (1.3) and ω an iid random potential satisfying $\mathbb{E}[\log(1 + |\omega_0|)] < \infty$. If for Lebesgue-almost all $E \in \Sigma_{\text{ac}}$ the cocycle equation associated with $\hat{H}(\theta)$:*

$$\Gamma(\theta) = \frac{1}{U(\theta) - E - \Gamma(S\theta)} \quad (2.4)$$

has not more than one measurable solution for a function $\Gamma : \Xi \rightarrow \mathbb{C}^+$ with $\text{Im } \Gamma \geq 0$, then the almost-sure AC spectrum of $\hat{H}_\lambda(\theta, \omega)$ is continuous at $\lambda = 0$, in the sense that for every Borel subset $I \subseteq \Sigma_{\text{ac}}$

$$\lim_{\lambda \rightarrow 0} \left| \sigma_{\text{ac}}(\hat{H}_\lambda(\theta, \omega)) \cap \sqrt{K}I \right| = \sqrt{K} |I|, \quad (2.5)$$

where $|\cdot|$ denotes Lebesgue measure.

Remark 2.5. The result proven in [1] includes also the statement that in a suitable sense the AC density of the spectral measure associated with the root vector is L^1 -continuous at $\lambda = 0$. Moreover, the derivation used a weaker condition than independence for the disorder variables ω . Rather, it suffices there to assume only weak correlations for the joint distributions along any two disjoint forward subtrees of \mathbb{T} . The results presented here are directly applicable also to this generalization.

To shed some light on (2.4), which is equivalent to (1.7), let us note that the Schrödinger equation $\phi_{n-1} + \phi_{n+1} + (V_n - E)\phi_n = 0$, can equivalently be expressed in terms of the Ricatti variables $\gamma_n := -\phi_n/\phi_{n-1}$ as:

$$\gamma_n = \frac{1}{V_n - E - \gamma_{n+1}}. \quad (2.6)$$

If the eigenstate state ϕ can be chosen as a covariant function of θ , then $\Gamma := \gamma_0[\phi]$ will satisfy (2.4). By a *covariant eigenstate* we mean here a generalized eigenfunction whose θ -dependence is such that

- (i) $H(\theta)\phi(\theta) = E\phi(\theta)$,
- (ii) $\phi_{n+1}(\theta) = e^{i\kappa(\theta)}\phi_n(S\theta)$ for some measurable $\kappa : \Xi \rightarrow \mathbb{R}$,

where (i) is to be taken in the weak sense, as appropriate for a generalized eigenfunction.

As an aside, let us note that the converse is also true. More precisely, if $\Gamma : \Xi \rightarrow \mathbb{C}^+$ is a solution of (2.4) with $\text{Im } \Gamma > 0$ then

$$\phi_{-1}(\theta) := \frac{1}{\sqrt{\text{Im } \Gamma(\theta)}}, \quad \phi_0(\theta) := \phi_{-1}(\theta)\Gamma(\theta) \quad (2.7)$$

determine a covariant eigenstate of $H(\theta)$, as was noted already in [8, 15].

The difference between a covariant eigenstate and a covariant eigenfunction is significant. States are interpreted as rays, for which the phase factor in (ii) has no effect. Covariant eigenfunctions, interpreted as solutions of (i) with $\kappa = 0$ in (ii) may not exist, or exist only at isolated energies. However, solutions with non-zero, constant κ do occur, and form the appropriate generalization of the Bloch-Floquet states to QP operators. We present their definition in the next section, where we also state our main result.

Remark 2.6. For the analysis of [1] it was relevant that the diagonal element of the Green function of $H^+(\theta)$ taken at the root 0,

$$\Gamma(E, \theta) = (H^+(\theta) - E - i0)^{-1}(0, 0), \quad (2.8)$$

is a solution of (2.4) with $\text{Im } \Gamma(E, \theta) > 0$ for almost every $E \in \sigma_{\text{ac}}(H(\theta))$. The proof of Proposition 2.4 proceeds by establishing that in any limit $\lambda, \eta \rightarrow 0$ the distribution of the diagonal Green function $\widehat{\Gamma}_\lambda(E + i\eta, \theta, \omega)$ of $\widehat{H}_\lambda(\theta, \omega)$ is of vanishing width, in the dependence on ω [1]. The subtle point is that the values may still depend on the way λ and η approach their limit. Any accumulation point satisfies the cocycle equation (2.4), but that in itself does not yet allow to conclude that it coincides with $K^{-1/2} \Gamma(E/\sqrt{K}, \theta)$. However, the uniqueness of solution which is discussed here guarantees the distributional convergence of $\widehat{\Gamma}_\lambda(E + i\eta, \theta, \omega)$, and by implication the continuity of the AC spectrum.

3. BLOCH-FLOQUET EIGENFUNCTIONS AND THE MAIN RESULT

Definition 1. A *Bloch-Floquet (BF) eigenfunction* of H with energy $E \in \mathbb{R}$ and quasi-momentum $k \in (-\pi, \pi]$ is a non-vanishing function $\psi : \mathbb{Z} \times \Xi \rightarrow \mathbb{C}$ with the properties:

- (i) $H(\theta) \psi(\theta) = E \psi(\theta)$ (in the weak sense).
- (ii) $\psi_n(\theta) = e^{ikn} \varphi(S^n \theta)$ for some continuous $\varphi : \Xi \rightarrow \mathbb{C}$.

We say that ψ forms part of a *conjugate BF pair* iff $\psi(\theta)$ and $\overline{\psi(\theta)}$ are linearly independent for almost every θ .

Remarks 3.1. (i) The second requirement in the above definition is equivalent to the covariance property

$$\psi_{n+1}(\theta) = e^{ik} \psi_n(S\theta). \quad (3.1)$$

(ii) Bloch-Floquet eigenfunctions come naturally in pairs. Namely, if ψ is one then its complex-conjugate $\overline{\psi}$ is also a BF eigenfunction with energy E and reversed quasi-momentum. Their Wronskian

$$[\psi, \overline{\psi}](\theta) := \psi_0(\theta) \overline{\psi_{-1}(\theta)} - \overline{\psi_0(\theta)} \psi_{-1}(\theta), \quad (3.2)$$

is independent of θ . Therefore, if $\psi(\theta)$ and $\overline{\psi(\theta)}$ are linearly independent for some θ , they are linearly independent for all θ and $[\psi, \overline{\psi}] \neq 0$. This implies that the ratio

$$\gamma(\theta) := -\frac{\psi_0(\theta)}{\psi_{-1}(\theta)} \quad (3.3)$$

is well-defined and takes values with either $\text{Im } \gamma(\theta) > 0$ or $\text{Im } \gamma(\theta) < 0$ for all θ .

Our main observation, whose proof is presented below, is that the existence of a conjugate BF pair of H at almost all energies in I allows one to conclude that the cocycle equation (2.4) has a unique solution and hence the criterion of Proposition 2.4 is met. That yields:

Theorem 3.2. *In the situation of Proposition 2.4 let $I \subseteq \Sigma_{\text{ac}}$ be a Borel set such that at Lebesgue-almost all $E \in I$, H admits a conjugate pair of Bloch-Floquet states. Then (2.5) holds.*

4. UNIQUENESS OF THE COVARIANT SOLUTION OF THE PROJECTIVE SCHRÖDINGER EQUATION

4.1. BF states and reducibility of the Schrödinger cocycle. More frequently, the discussion of the Schrödinger cocycle is carried out in the space $\Xi \times SL(2, \mathbb{R})$. The role of $\mathcal{A}(E, \theta)$ in (1.5) is taken by the matrix

$$A(E, \theta) := \begin{pmatrix} E - U(\theta) & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.1)$$

In this notation, the existence of BF pairs is known to be equivalent to the *reducibility* of this cocycle to a constant one:

Proposition 4.1 (cf. [21]). *The following two statements are equivalent:*

- (i) *H admits a conjugate BF pair with energy E and quasi-momenta $\pm k$.*
- (ii) *the Schrödinger cocycle is reducible to a constant matrix,*

$$Z(S\theta)^{-1} A(E, \theta) Z(\theta) = \begin{pmatrix} e^{-ik} & 0 \\ 0 & e^{ik} \end{pmatrix} \quad (4.2)$$

where $Z(\theta) =: \begin{pmatrix} \overline{\psi_0(\theta)} & \psi_0(\theta) \\ \overline{\psi_{-1}(\theta)} & \psi_{-1}(\theta) \end{pmatrix}$ is not singular, i.e., $\det Z(\theta) \neq 0$.

Remark 4.2. If H admits a conjugate pair of covariant eigenstates, which according to (2.7) holds for almost every $E \in \sigma_{ac}(H(\theta))$, then (4.2) remains true if k is replaced by $\kappa(\theta)$.

Historically, reducibility was first established for various examples of QP operators H for almost all energies in their AC spectra [10, 5, 23, 7, 11, 19, 12, 13, 6, 21]. In particular, it was proven to hold for the almost Mathieu operator and the family of real analytic QP potentials mentioned in Remarks 2.2(ii) and 2.2(iii). In this context, it was natural to ask whether all quasi-periodic operators admit conjugate BF pairs for almost every $E \in \Sigma_{ac}$. This question was raised in [8], and was disproved in [16] in the general case. However, for shifts S on the one-dimensional torus with certain Diophantine frequencies α and arbitrarily often differentiable U , it was shown in [3] that BF pairs indeed exist for almost every $E \in \Sigma_{ac}$.

4.2. Reducibility and uniqueness. The main observation leading to Theorem 3.2 is:

Lemma 4.3. *Assume that H admits a conjugate BF pair with energy E and quasi-momenta $\pm k$ with $|k|/\pi \notin \{(m \cdot \alpha) \bmod \mathbb{Z} : m \in \mathbb{Z}^\nu\}$. Then (1.7) has a unique solution with values in \mathbb{C}^+ .*

Remark 4.4. The above condition on the quasi-momenta can be rephrased in terms of the integrated density of states (IDS) $n(E)$ of $H(\theta)$ (see [4] for a definition) using its relation to the rotation number of the Schrödinger cocycle [9]. Namely, there exists some $m \in \mathbb{Z}^\nu$ such that

$$\frac{|k|}{2\pi} = \left(\frac{n(E)}{2} + m \cdot \alpha \right) \bmod \mathbb{Z}. \quad (4.3)$$

Proof of Lemma 4.3. The Möbius transformation $\mathcal{A}(E, \theta)$ given by (1.6) is just a projective counterpart of $A(E, \theta)$ in (4.1). From Proposition 4.1 it thus follows that $\mathcal{A}(E, \theta)$ is reducible to a constant cocycle, in the sense that there exists a Möbius mapping $\mathcal{Z}(\theta)$ such that

$$\mathcal{A}(E, \theta) \mathcal{Z}(\theta) = \mathcal{Z}(S\theta) e^{-2ik}. \quad (4.4)$$

An explicit expression for $\mathcal{Z}(\theta)$, in terms of the BF state $\psi(\theta)$, is

$$\mathcal{Z}(\theta) : \gamma \mapsto \gamma = \frac{\overline{\psi_0(\theta)} \gamma - \psi_0(\theta)}{-\overline{\psi_{-1}(\theta)} \gamma + \psi_{-1}(\theta)}. \quad (4.5)$$

As was noted before, $\gamma(\theta) := -\psi_0(\theta)/\psi_{-1}(\theta)$ satisfies (1.7) and according to Remark 3.1(ii) we may assume without loss of generality that $\text{Im } \gamma(\theta) > 0$ for all θ . Suppose now there exists yet another solution $\tilde{\gamma} \neq \gamma$ of (1.7) with values in \mathbb{C}^+ . Then

$$f(\theta) := \mathcal{Z}(\theta)^{-1} \tilde{\gamma}(\theta) = \frac{\psi_{-1}(\theta)}{\overline{\psi_{-1}(\theta)}} \frac{\tilde{\gamma}(\theta) - \gamma(\theta)}{\overline{\tilde{\gamma}(\theta) - \gamma(\theta)}} \quad (4.6)$$

satisfies

$$(\mathcal{S}f)(\theta) := f(S\theta) = \exp[-2ik] f(\theta) \quad (4.7)$$

where we have introduced the unitary Koopmann operator $\mathcal{S} : L^2(\Xi) \rightarrow L^2(\Xi)$ associated with the ergodic shift S . For a QP shift with frequency α its spectrum consists of the countable set

$$\text{spec}(\mathcal{S}) = \{ \exp(2\pi i m \cdot \alpha) : m \in \mathbb{Z}^\nu \}. \quad (4.8)$$

According to (4.7), f is a proper eigenfunction of the Koopmann operator \mathcal{S} . Since $\exp[\pm 2i|k|] \notin \text{spec}(\mathcal{S})$ this implies that $f = 0$, which is a contradiction. $\square \quad \square$

5. PROOF OF THE MAIN RESULT

We are now ready to complete the prove of our main result.

Proof of Theorem 3.2. In view of Proposition 2.4, Lemma 4.3 and Remark 4.4 it remains to show that for almost all $E \in \Sigma_{\text{ac}}$

$$n(E) \notin \{ (m \cdot \alpha) \bmod \mathbb{Z} : m \in \mathbb{Z}^\nu \}. \quad (5.1)$$

But this follows from the fact that Σ_{ac} is a set of positive Lebesgue measure on which the integrated density of states is not constant [4]. Since the right side in (5.1) is a countable set, it cannot coincide with the image of Σ_{ac} under the map $E \mapsto n(E)$. $\square \quad \square$

Remark 5.1. By the gap-labeling theorem [14, 9] the condition $n(E) = (m \cdot \alpha) \bmod \mathbb{Z}$, for some $m \in \mathbb{Z}^\nu$, characterizes the spectral gaps of $H(\theta)$.

ACKNOWLEDGEMENT

We thank Robert Sims, Svetlana Jitomirskaya, Michael Goldstein and Uzy Smilansky for stimulating discussions of topics related to this work. MA thanks for the gracious hospitality enjoyed at the Weizmann Institute. This work was supported in parts by the Einstein Center for Theoretical Physics and the Minerva Center for Nonlinear Physics at the Weizmann Institute, by the US National Science Foundation, and by the Deutsche Forschungsgemeinschaft.

REFERENCES

- [1] M. Aizenman, R. Sims and S. Warzel. Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs. Preprint math-ph/0502006. To appear in *Prob. Theor. Relat. Fields*
- [2] M. Aizenman, R. Sims and S. Warzel. Absolutely continuous spectra of quantum tree graphs with weak disorder. *Comm. Math. Phys.* 264:371–389, 2006.
- [3] A. Avila and R. Krikorian. Reducibility or non-uniform hyperbolicity for quasiperiodic Schrödinger cocycles. Preprint math.DS/0306382. To appear in *Ann. Math.*
- [4] J. Avron and B. Simon. Almost periodic Schrödinger operators II. The integrated density of states. *Duke Math. J.* 50:369–391, 1983.

- [5] J. Bellissard, R. Lima and D. Testard. A metal-insulator transition for the almost Mathieu model. *Comm. Math. Phys.* 88:207–234, 1983.
- [6] J. Bourgain and S. Jitomirskaya. Absolutely continuous spectrum for 1D quasiperiodic operators. *Invent. Math.* 148:453–463, 2002.
- [7] V. Chulaevsky and F. Delyon. Purely absolutely continuous spectrum for almost Mathieu operators. *J. Stat. Phys.* 55:1279–1284, 1989.
- [8] P. Deift and B. Simon. Almost periodic Schrödinger operators III. The absolutely continuous spectrum in one dimension. *Comm. Math. Phys.* 90:389–411, 1983.
- [9] F. Delyon and B. Souillard. The rotation number for finite difference operators and its properties. *Comm. Math. Phys.* 146:447–482, 1992.
- [10] E. Dinaburg and Y. Sinai. The one-dimensional Schrödinger equation with quasi-periodic potential. *Funkt. Anal. i. Priloz.* 9:8–21, 1995.
- [11] L. H. Eliasson. Floquet solutions for the 1-dimensional quasiperiodic Schrödinger equation. *Comm. Math. Phys.* 146:447–482, 1992.
- [12] A. Y. Gordon, S. Jitomirskaya, Y. Last and B. Simon. Duality and singular continuous spectrum in the almost Mathieu equation. *Acta Math.* 178:169–183, 1997.
- [13] S. Jitomirskaya. Metal-insulator transition for the almost Mathieu operator. *Ann. Math.* 150:1159–1175, 1999.
- [14] R. Johnson and J. Moser. The rotation number for almost periodic potentials. *Comm. Math. Phys.* 84:403–438, 1982.
- [15] S. Kotani. Ljapunov indices determine absolute continuous spectra of stationary one dimensional Schrödinger operators. In K. Ito, editor, *Proc. Taniguchi Intern. Symp. on Stochastic Analysis*, pages 225–247, Amsterdam, 1983. North Holland.
- [16] Y. Last. A relation between a.c. spectrum of ergodic Jacobi matrices and the spectra of periodic approximants. *Comm. Math. Phys.* 151:183–192, 1993.
- [17] Y. Last and B. Simon. Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators. *Invent. Math.* 151:329–367, 1999.
- [18] L. Pastur and A. Figotin. *Spectra of random and almost-periodic operators*. Springer, Berlin, 1992.
- [19] J. Puig. Cantor spectrum for the almost Mathieu operator. *Comm. Math. Phys.* 244:297–309, 2004.
- [20] J. Puig. *Reducibility of quasi-periodic skew-products and the spectrum of Schrödinger operators*. PhD Thesis, University of Barcelona, 2004.
- [21] J. Puig. A non-perturbative Eliasson’s reducibility theorem. Preprint math.DS/0503356.
- [22] B. Simon. Kotani theory for one-dimensional Jacobi matrices. *Comm. Math. Phys.*, 89:227–234, 1983.
- [23] Y. G. Sinai. Anderson localization for one-dimensional difference Schrödinger operator with quasiperiodic potential. *J. Stat. Phys.* 46:861–909, 1987.

DEPARTMENTS OF MATHEMATICS AND PHYSICS, PRINCETON UNIVERSITY, PRINCETON NJ 08544, USA.

E-mail address: aizenman@princeton.edu

E-mail address: swarzel@princeton.edu